

SOME q -EXPONENTIAL FORMULAS FOR FINITE-DIMENSIONAL \square_q -MODULES

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ABSTRACT. We consider the algebra \square_q which is a mild generalization of the quantum algebra $U_q(\mathfrak{sl}_2)$. The algebra \square_q is defined by generators and relations. The generators are $\{x_i\}_{i \in \mathbb{Z}_4}$, where \mathbb{Z}_4 is the cyclic group of order 4. For $i \in \mathbb{Z}_4$ the generators x_i, x_{i+1} satisfy a q -Weyl relation, and x_i, x_{i+2} satisfy a cubic q -Serre relation. For $i \in \mathbb{Z}_4$ we show that the action of x_i is invertible on each nonzero finite-dimensional \square_q -module. We view x_i^{-1} as an operator that acts on nonzero finite-dimensional \square_q -modules. For $i \in \mathbb{Z}_4$, define $n_{i,i+1} = q(1 - x_i x_{i+1})/(q - q^{-1})$. We show that the action of $n_{i,i+1}$ is nilpotent on each nonzero finite-dimensional \square_q -module. We view the q -exponential $\exp_q(n_{i,i+1})$ as an operator that acts on nonzero finite-dimensional \square_q -modules. In our main results, for $i, j \in \mathbb{Z}_4$ we express each of $\exp_q(n_{i,i+1})x_j \exp_q(n_{i,i+1})^{-1}$ and $\exp_q(n_{i,i+1})^{-1}x_j \exp_q(n_{i,i+1})$ as a polynomial in $\{x_k^{\pm 1}\}_{k \in \mathbb{Z}_4}$.

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1. INTRODUCTION

This paper is about a certain algebra \square_q ; we will recall the definition shortly. Broadly speaking it can be viewed as a generalization of the quantum algebra $U_q(\mathfrak{sl}_2)$. In order to motivate our results we make some comments about $U_q(\mathfrak{sl}_2)$. We will work with the equitable presentation, which was introduced in [9] and investigated further in [1–3, 5–8, 11–14, 16]. Let \mathbb{F} denote an algebraically closed field. Fix $0 \neq q \in \mathbb{F}$ that is not a root of unity. In the equitable presentation, the \mathbb{F} -algebra $U_q(\mathfrak{sl}_2)$ has generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = y^{-1}y = 1$,

$$\frac{qxy - qyx}{q - q^{-1}} = 1, \quad \frac{qyz - qzy}{q - q^{-1}} = 1, \quad \frac{qzx - qxz}{q - q^{-1}} = 1.$$

Define

$$n_x = \frac{q(1 - yz)}{q - q^{-1}}, \quad n_y = \frac{q(1 - zx)}{q - q^{-1}}, \quad n_z = \frac{q(1 - xy)}{q - q^{-1}}.$$

On each nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module, each of x, y, z is invertible (see [9, Corollary 4.5]) and each of n_x, n_y, n_z is nilpotent (see [9, Lemma 5.5]). Recall from [10, p. 204] the q -exponential function

$$\exp_q(T) = \sum_{n \in \mathbb{N}} \frac{q^{\binom{n}{2}}}{[n]_q!} T^n.$$

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In [9, Sections 5, 6] it was shown that the following equations hold on each nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module:

$$\begin{aligned} \exp_q(n_x)x \exp_q(n_x)^{-1} &= x + z - z^{-1}, \\ \exp_q(n_x)y \exp_q(n_x)^{-1} &= z^{-1}, \\ \exp_q(n_x)z \exp_q(n_x)^{-1} &= zyz, \\ \exp_q(n_x)^{-1}x \exp_q(n_x) &= x + y - y^{-1}, \\ \exp_q(n_x)^{-1}y \exp_q(n_x) &= yzy, \\ \exp_q(n_x)^{-1}z \exp_q(n_x) &= y^{-1}. \end{aligned}$$

Cyclically permuting x, y, z in the above equations, we get 12 more equations. Our goal in this paper is to find analogous equations that apply to \square_q .

We now discuss the algebra \square_q . This algebra was introduced in [15, Definition 5.1]. The main result of [15] is an injective algebra homomorphism from the q -Onsager algebra \mathcal{O}_q to \square_q . By [15, Proposition 5.5] and [4, Propositions 4.1, 4.3], there exists an injective algebra homomorphism from \square_q to the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$.

The \mathbb{F} -algebra \square_q is defined as follows (formal definitions start in Section 2). The generators are $\{x_i\}_{i \in \mathbb{Z}_4}$, where $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ is the cyclic group of order 4. The relations are

$$\begin{aligned} \frac{qx_i x_{i+1} - q^{-1}x_{i+1}x_i}{q - q^{-1}} &= 1, \\ x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 &= 0, \end{aligned}$$

for $i \in \mathbb{Z}_4$. We show that the action of each x_i is invertible on each nonzero finite-dimensional \square_q -module. We view x_i^{-1} as an operator that acts on nonzero finite-dimensional \square_q -modules. For $i \in \mathbb{Z}_4$ define

$$\mathbf{n}_{i,i+1} = \frac{q(1 - x_i x_{i+1})}{q - q^{-1}}.$$

We show that the action of $\mathbf{n}_{i,i+1}$ is nilpotent on each nonzero finite-dimensional \square_q -module. We view the q -exponential $\exp_q(\mathbf{n}_{i,i+1})$ as an operator that acts on nonzero finite-dimensional \square_q -modules. For $i, j \in \mathbb{Z}_4$ consider the two expressions

$$(1.1) \quad \exp_q(\mathbf{n}_{i,i+1})x_j \exp_q(\mathbf{n}_{i,i+1})^{-1}, \quad \exp_q(\mathbf{n}_{i,i+1})^{-1}x_j \exp_q(\mathbf{n}_{i,i+1}).$$

In our main results we express each of (1.1) as a polynomial in $\{x_k^{\pm 1}\}_{k \in \mathbb{Z}_4}$. These results are Propositions 8.1, 8.2 and Propositions 9.3–9.6.

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 contains some basic facts about \square_q . In Section 4 we describe some isomorphisms and antiisomorphisms for \square_q . In Section 5 we show that the action of each x_i is invertible on each nonzero finite-dimensional \square_q -module. In Section 6 we show that the action of each $\mathbf{n}_{i,i+1}$ is nilpotent on each finite-dimensional \square_q -module. In Section 7 we review the q -exponential function, and apply it to $\mathbf{n}_{i,i+1}$. In Sections 8 and 9 we prove our main results.

2. PRELIMINARIES

Throughout the paper, we fix the following notation. Let \mathbb{F} denote an algebraically closed field. Recall the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and the

ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. We will be discussing algebras. An algebra is meant to be associative and have a 1.

Let V denote a finite-dimensional vector space over \mathbb{F} . Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . An element $A \in \text{End}(V)$ is called *nilpotent* whenever there exists a positive integer n such that $A^n = 0$. By an *eigenvalue* of A , we mean a root of the characteristic polynomial of A .

Fix $0 \neq q \in \mathbb{F}$ such that q is not a root of unity. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

For $n \in \mathbb{N}$ define

$$[n]_q! = \prod_{i=1}^n [i]_q.$$

We interpret $[0]_q! = 1$.

3. THE ALGEBRA \square_q

In this section, we recall the algebra \square_q .

Definition 3.1. [15, Definition 5.1] Let \square_q denote the \mathbb{F} -algebra with generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$(3.1) \quad \frac{qx_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1,$$

$$(3.2) \quad x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0.$$

The structure of the algebra \square_q is analyzed in [15]. We don't need the full strength of the results in [15], but we will use the following fact.

Lemma 3.2. *The elements $\{x_i\}_{i \in \mathbb{Z}_4}$ are linearly independent in \square_q .*

Proof. By [15, Proposition 5.5]. □

We now give some formulas for later use.

Lemma 3.3. *For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$ the following relations hold in \square_q :*

$$(3.3) \quad q^n x_i^n x_{i+1} - q^{-n} x_{i+1} x_i^n = (q^n - q^{-n}) x_i^{n-1},$$

$$(3.4) \quad q^n x_i x_{i+1}^n - q^{-n} x_{i+1}^n x_i = (q^n - q^{-n}) x_{i+1}^{n-1}.$$

Proof. Use (3.1) and induction on n . □

Lemma 3.4. *For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relation holds in \square_q :*

$$(3.5) \quad \begin{aligned} x_i^n x_{i+2} &= \frac{[n-1]_q [n-2]_q}{[2]_q} x_{i+2} x_i^n - [n]_q [n-2]_q x_i x_{i+2} x_i^{n-1} \\ &\quad + \frac{[n]_q [n-1]_q}{[2]_q} x_i^2 x_{i+2} x_i^{n-2}. \end{aligned}$$

Proof. Use (3.2) and induction on n . □

For $i \in \mathbb{Z}_4$ we define an element $\mathbf{n}_{i,i+1} \in \square_q$. Later we will show that $\mathbf{n}_{i,i+1}$ is nilpotent on each finite-dimensional \square_q -module.

Lemma 3.5. For $i \in \mathbb{Z}_4$,

$$q(1 - x_i x_{i+1}) = q^{-1}(1 - x_{i+1} x_i).$$

Proof. This equation is a reformulation of (3.1). \square

Definition 3.6. For $i \in \mathbb{Z}_4$ define

$$(3.6) \quad \mathbf{n}_{i,i+1} = \frac{q(1 - x_i x_{i+1})}{q - q^{-1}} = \frac{q^{-1}(1 - x_{i+1} x_i)}{q - q^{-1}}.$$

We now describe some basic properties of $\mathbf{n}_{i,i+1}$ for later use.

Lemma 3.7. For $i \in \mathbb{Z}_4$, the following relations hold in \square_q :

$$(3.7) \quad x_i \mathbf{n}_{i,i+1} = q^{-2} \mathbf{n}_{i,i+1} x_i, \quad x_{i+1} \mathbf{n}_{i,i+1} = q^2 \mathbf{n}_{i,i+1} x_{i+1}.$$

Proof. To verify (3.7), eliminate $\mathbf{n}_{i,i+1}$ using the first equality in (3.6) and simplify the result using (3.1). \square

Corollary 3.8. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relations hold in \square_q :

$$(3.8) \quad x_i^n \mathbf{n}_{i,i+1} = q^{-2n} \mathbf{n}_{i,i+1} x_i^n, \quad x_{i+1}^n \mathbf{n}_{i,i+1} = q^{2n} \mathbf{n}_{i,i+1} x_{i+1}^n.$$

Proof. By (3.7) and induction on n . \square

Lemma 3.9. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relations hold in \square_q :

$$(3.9) \quad x_i^n x_{i+1}^n \left(1 - (q^{-2n} - q^{-2n-2}) \mathbf{n}_{i,i+1} \right) = x_i^{n+1} x_{i+1}^{n+1},$$

$$(3.10) \quad \left(1 - (q^{2n+2} - q^{2n}) \mathbf{n}_{i,i+1} \right) x_{i+1}^n x_i^n = x_{i+1}^{n+1} x_i^{n+1}.$$

Proof. In order to verify these equations, eliminate $\mathbf{n}_{i,i+1}$ using the first equality in (3.6) and simplify the result using (3.4). \square

4. SOME ISOMORPHISMS AND ANTIISOMORPHISMS FOR \square_q

In this section, we introduce some isomorphisms and antiisomorphisms for \square_q . By an *automorphism* of \square_q we mean an \mathbb{F} -algebra isomorphism from \square_q to \square_q .

Lemma 4.1. There exists an automorphism ρ of \square_q that sends $x_i \mapsto x_{i+1}$ for $i \in \mathbb{Z}_4$.

Proof. By Definition 3.1. \square

Lemma 4.2. The map ρ from Lemma 4.1 sends $\mathbf{n}_{i,i+1} \mapsto \mathbf{n}_{i+1,i+2}$ for $i \in \mathbb{Z}_4$.

Proof. By the definition of ρ and Definition 3.6. \square

We recall the notion of antiisomorphism. Given \mathbb{F} -algebras \mathcal{A}, \mathcal{B} , a map $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ is called an *antiisomorphism* whenever γ is an isomorphism of \mathbb{F} -vector spaces and $(ab)^\gamma = b^\gamma a^\gamma$ for all $a, b \in \mathcal{A}$. An antiisomorphism $\gamma : \mathcal{A} \rightarrow \mathcal{A}$ is called an *antiautomorphism* of \mathcal{A} .

Lemma 4.3. There exists an antiautomorphism ϕ of \square_q that sends

$$x_0 \leftrightarrow x_1, \quad x_2 \leftrightarrow x_3.$$

Moreover there exists an antiautomorphism φ of \square_q that sends

$$x_1 \leftrightarrow x_2, \quad x_3 \leftrightarrow x_0.$$

Proof. By Definition 3.1. \square

Lemma 4.4. *The map ϕ from Lemma 4.3 sends*

$$\mathbf{n}_{0,1} \mapsto \mathbf{n}_{0,1}, \quad \mathbf{n}_{2,3} \mapsto \mathbf{n}_{2,3}, \quad \mathbf{n}_{1,2} \leftrightarrow \mathbf{n}_{3,0}.$$

Moreover the map φ from Lemma 4.3 sends

$$\mathbf{n}_{1,2} \mapsto \mathbf{n}_{1,2}, \quad \mathbf{n}_{3,0} \mapsto \mathbf{n}_{3,0}, \quad \mathbf{n}_{0,1} \leftrightarrow \mathbf{n}_{2,3}.$$

Proof. By the definitions of ϕ, φ and Definition 3.6. \square

Lemma 4.5. *The maps ρ from Lemma 4.1 and ϕ, φ from Lemma 4.3 satisfy the following relations:*

$$(4.1) \quad \rho^4 = \phi^2 = \varphi^2 = (\rho\phi)^2 = (\rho\varphi)^2 = 1,$$

$$(4.2) \quad \rho\phi = \varphi\rho, \quad \rho\varphi = \phi\rho.$$

Proof. By the definitions of ρ, ϕ, φ . \square

Recall that the dihedral group D_4 has the following group presentation:

$$D_4 = \{x, y \mid x^4 = y^2 = (xy)^2 = 1\}.$$

The group D_4 has 8 elements and is the group of symmetries of a square. Consider the group $\text{AAut}(\square_q)$ consisting of the automorphisms and antiautomorphisms of \square_q . The group operation is composition.

Lemma 4.6. *Let G denote the subgroup of $\text{AAut}(\square_q)$ generated by the maps ρ from Lemma 4.1 and ϕ, φ from Lemma 4.3. Then G is isomorphic to D_4 .*

Proof. By (4.1) there exists a group homomorphism $f : D_4 \rightarrow G$ that sends $x \mapsto \rho$ and $y \mapsto \phi$. By (4.2) the element φ is in the image of f . Therefore f is surjective. By Lemma 3.2, the map f is injective. By these comments f is an isomorphism. The result follows. \square

We now relate \square_q and $\square_{q^{-1}}$.

Lemma 4.7. *There exists an antiisomorphism $\dagger : \square_q \rightarrow \square_{q^{-1}}$ that sends $x_i \mapsto x_i$ for $i \in \mathbb{Z}_4$. Moreover $\dagger^2 = 1$.*

Proof. By Definition 3.1. \square

In Definition 3.6 we discussed an element $\mathbf{n}_{i,i+1} \in \square_q$. We retain the notation $\mathbf{n}_{i,i+1}$ for the corresponding element in $\square_{q^{-1}}$.

Lemma 4.8. *The map \dagger from Lemma 4.7 sends $\mathbf{n}_{i,i+1} \mapsto -\mathbf{n}_{i,i+1}$ for $i \in \mathbb{Z}_4$.*

Proof. By the definition of \dagger and Definition 3.6. \square

5. THE ELEMENT x_i IS INVERTIBLE ON FINITE-DIMENSIONAL \square_q -MODULES

Let V denote a nonzero finite-dimensional \square_q -module. In this section, we show that for $i \in \mathbb{Z}_4$ the action of x_i on V is invertible. We first show that the action of x_i on V is not nilpotent.

Lemma 5.1. *Let V denote a nonzero finite-dimensional \square_q -module. For $i \in \mathbb{Z}_4$, the action of x_i on V is not nilpotent.*

Proof. Assume that x_i is nilpotent on V . Then there exists a minimal positive integer n such that $x_i^n = 0$ on V . By (3.1), we have $n \neq 1$. By (3.3) and since q is not a root of unity, we have $x_i^{n-1} = 0$ on V . This contradicts the minimality of n . The result follows. \square

We will use the following notation. Let V denote a finite-dimensional vector space over \mathbb{F} and let $A \in \text{End}(V)$. For $\theta \in \mathbb{F}$ define

$$V_A(\theta) = \{v \in V \mid \exists n \in \mathbb{N}, (A - \theta I)^n v = 0\}.$$

Observe that θ is an eigenvalue of A if and only if $V_A(\theta) \neq 0$, and in this case $V_A(\theta)$ is the corresponding generalized eigenspace. The sum $V = \sum_{\theta \in \mathbb{F}} V_A(\theta)$ is direct.

Proposition 5.2. *Let V denote a nonzero finite-dimensional \square_q -module. For $i \in \mathbb{Z}_4$ the action of x_i on V is invertible.*

Proof. To show x_i is invertible on V , it suffices to show that 0 is not an eigenvalue of x_i . Consider the subspace $W = V_{x_i}(0)$. We first show that W is \square_q -invariant. By construction, W is x_i -invariant. Pick $v \in W$. By the definition of W , there exists $m \in \mathbb{N}$ such that $x_i^m v = 0$. By (3.3) with $n-1 = m$, we have $x_i^{m+1} x_{i+1} v = 0$. Therefore $x_{i+1} v \in W$. By (3.4) with $n-1 = m$, we have $x_i^{m+1} x_{i+3} v = 0$. Therefore $x_{i+3} v \in W$. By (3.5) with $n-2 = m$, we have $x_i^{m+2} x_{i+2} v = 0$. Therefore $x_{i+2} v \in W$. We have shown that W is invariant under x_j for $j \in \mathbb{Z}_4$. Therefore W is \square_q -invariant. By construction x_i is nilpotent on W . Therefore $W = 0$ in view of Lemma 5.1. The result follows. \square

Motivated by Proposition 5.2, we make the following definition.

Definition 5.3. For $i \in \mathbb{Z}_4$, let x_i^{-1} denote the operator that acts on each nonzero finite-dimension \square_q -module as the inverse of x_i .

We now give some formulas involving the operators x_i^{-1} .

Lemma 5.4. *For $i \in \mathbb{Z}_4$ the following relations hold on each nonzero finite-dimension \square_q -module:*

$$(5.1) \quad qx_{i+1}x_i^{-1} - q^{-1}x_i^{-1}x_{i+1} = (q - q^{-1})x_i^{-2},$$

$$(5.2) \quad qx_{i+1}^{-1}x_i - q^{-1}x_i x_{i+1}^{-1} = (q - q^{-1})x_{i+1}^{-2}.$$

Proof. By (3.1) and Definition 5.3. \square

Lemma 5.5. *For $i \in \mathbb{Z}_4$ the following relations hold on each nonzero finite-dimension \square_q -module:*

$$(5.3) \quad \frac{qx_i^{-2}x_{i+1}^{-1} + q^{-1}x_{i+1}^{-1}x_i^{-2}}{q + q^{-1}} = x_{i+1}^{-1}x_i^{-1}x_{i+1}x_i^{-1}x_{i+1}^{-1},$$

$$(5.4) \quad \frac{qx_i^{-1}x_{i+1}^{-2} + q^{-1}x_{i+1}^{-2}x_i^{-1}}{q + q^{-1}} = x_i^{-1}x_{i+1}^{-1}x_i x_{i+1}^{-1}x_i^{-1}.$$

Proof. We first show (5.3). In (5.1) multiply each term on the left by x_{i+1}^{-1} and on the right by $x_i^{-1}x_{i+1}^{-1}$ to get

$$(5.5) \quad qx_i^{-2}x_{i+1}^{-1} - q^{-1}x_{i+1}^{-1}x_i^{-1}x_{i+1}x_i^{-1}x_{i+1}^{-1} = (q - q^{-1})x_{i+1}^{-1}x_i^{-3}x_{i+1}^{-1}.$$

Similarly in (5.1), multiply each term on the left by $x_{i+1}^{-1}x_i^{-1}$ and on the right by x_{i+1}^{-1} to get

$$(5.6) \quad qx_{i+1}^{-1}x_i^{-1}x_{i+1}x_i^{-1}x_{i+1}^{-1} - q^{-1}x_{i+1}^{-1}x_i^{-2} = (q - q^{-1})x_{i+1}^{-1}x_i^{-3}x_{i+1}^{-1}.$$

Subtract (5.5) from (5.6) and solve for $x_{i+1}^{-1}x_i^{-1}x_{i+1}x_i^{-1}x_{i+1}^{-1}$ to get (5.3). To get (5.4), apply the map ϕ from Lemma 4.3 to each side of (5.3). \square

6. THE ELEMENT $\mathbf{n}_{i,i+1}$ IS NILPOTENT ON FINITE-DIMENSIONAL \square_q -MODULES

Let V denote a finite-dimensional \square_q -module. In this section, we show that for $i \in \mathbb{Z}_4$ the action of $\mathbf{n}_{i,i+1}$ on V is nilpotent.

Lemma 6.1. *Let V denote a finite-dimensional \square_q -module and let $\theta \in \mathbb{F}$. Then for $i \in \mathbb{Z}_4$, we have $\mathbf{n}_{i,i+1}V_{x_i}(\theta) \subseteq V_{x_i}(q^{-2}\theta)$.*

Proof. Pick $v \in V_{x_i}(\theta)$. We show $\mathbf{n}_{i,i+1}v \in V_{x_i}(q^{-2}\theta)$. By the definition of $V_{x_i}(\theta)$, there exists $n \in \mathbb{N}$ such that $(x_i - \theta I)^n v = 0$. By this and the left equation in (3.8), we have $(x_i - q^{-2}\theta I)^n \mathbf{n}_{i,i+1}v = 0$. Therefore $\mathbf{n}_{i,i+1}v \in V_{x_i}(q^{-2}\theta)$. The result follows. \square

Proposition 6.2. *Let V denote a finite-dimensional \square_q -module. For $i \in \mathbb{Z}_4$ the action of $\mathbf{n}_{i,i+1}$ on V is nilpotent.*

Proof. Assume that V is nonzero, otherwise the result is trivial. It suffices to show that for each eigenvalue θ of x_i , there exists a positive integer m such that $\mathbf{n}_{i,i+1}^m V_{x_i}(\theta) = 0$. By Proposition 5.2 the scalar 0 is not an eigenvalue of x_i . Therefore $\theta \neq 0$. Since V has finite positive dimension and q is not a root of unity, there exists a positive integer m such that θq^{-2j} is an eigenvalue of x_i for $0 \leq j \leq m-1$, but θq^{-2m} is not an eigenvalue of x_i . By this and Lemma 6.1, we have $\mathbf{n}_{i,i+1}^m V_{x_i}(\theta) \subseteq V_{x_i}(\theta q^{-2m}) = 0$. Therefore $\mathbf{n}_{i,i+1}^m V_{x_i}(\theta) = 0$. The result follows. \square

7. THE q -EXPONENTIAL FUNCTION

In this section we obtain some results involving the q -exponential function.

Definition 7.1. [10, p. 204] Let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\psi \in \text{End}(V)$ be nilpotent. Define

$$(7.1) \quad \exp_q(\psi) = \sum_{n \in \mathbb{N}} \frac{q^{\binom{n}{2}}}{[n]_q!} \psi^n.$$

The following result is well known and readily verified.

Lemma 7.2. [10, p. 204] Referring to Definition 7.1, the map $\exp_q(\psi)$ is invertible and its inverse is

$$\exp_{q^{-1}}(-\psi) = \sum_{n \in \mathbb{N}} \frac{(-1)^n q^{-\binom{n}{2}}}{[n]_q!} \psi^n.$$

We mention an identity for later use.

Lemma 7.3. *Referring to Definition 7.1,*

$$(7.2) \quad \exp_q(q^2\psi)(1 - (q^2 - 1)\psi) = \exp_q(\psi).$$

Proof. To verify (7.2), express each side as a power series in ψ using (7.1). \square

Pick $i \in \mathbb{Z}_4$. By Proposition 6.2, the action of $\mathbf{n}_{i,i+1}$ on each nonzero finite-dimensional \square_q -module is nilpotent. By this we view $\exp_q(\mathbf{n}_{i,i+1})$ as the operator that acts on each nonzero finite-dimensional \square_q -module. For $i, j \in \mathbb{Z}_4$, consider the following two expressions:

$$(7.3) \quad \exp_q(\mathbf{n}_{i,i+1})x_j\exp_q(\mathbf{n}_{i,i+1})^{-1}, \quad \exp_q(\mathbf{n}_{i,i+1})^{-1}x_j\exp_q(\mathbf{n}_{i,i+1}).$$

For each expression in (7.3), expand both q -exponential terms using Definition 7.1 and Lemma 7.2. This yields a double sum with infinitely many terms. We will show that in fact, each double sum is a polynomial in $\{x_k^{\pm 1}\}_{k \in \mathbb{Z}_4}$. We now give some formulas for later use.

Lemma 7.4. *For $i \in \mathbb{Z}_4$ and $r \in \mathbb{Z}$, the following relations hold on each nonzero finite-dimension \square_q -module:*

$$(7.4) \quad \exp_q(q^{2r}\mathbf{n}_{i,i+1}) = \exp_q(\mathbf{n}_{i,i+1})x_i^{-r}x_{i+1}^{-r},$$

$$(7.5) \quad \exp_q(q^{2r}\mathbf{n}_{i,i+1}) = x_i^{-r}x_{i+1}^{-r}\exp_q(\mathbf{n}_{i,i+1}).$$

Proof. To show (7.4) for $r \geq 0$, use induction on r . The calculation is routine using (7.2) with $\psi = q^{2r}\mathbf{n}_{i,i+1}$ along with (3.10). We similarly show (7.4) for $r < 0$ by induction on $r = -1, -2, \dots$ using (3.9) and (7.2). To get (7.5), apply the map ϕ from Lemma 4.3 to each side of (7.4). \square

8. SOME q -EXPONENTIAL FORMULAS, PART I

In this section, we analyze (7.3) for the case $j = i$ or $j = i + 1$.

Proposition 8.1. *For $i \in \mathbb{Z}_4$, the following relations hold on each nonzero finite-dimension \square_q -module:*

$$(8.1) \quad \exp_q(\mathbf{n}_{i,i+1})x_i\exp_q(\mathbf{n}_{i,i+1})^{-1} = x_{i+1}^{-1},$$

$$(8.2) \quad \exp_q(\mathbf{n}_{i,i+1})^{-1}x_{i+1}\exp_q(\mathbf{n}_{i,i+1}) = x_i^{-1}.$$

Proof. We first verify (8.2). By the equation on the right in (3.8) and Definition 7.1, we have

$$x_{i+1}\exp_q(\mathbf{n}_{i,i+1})x_{i+1}^{-1} = \exp_q(q^2\mathbf{n}_{i,i+1}).$$

Using this and (7.4) with $r = 1$ we routinely obtain (8.2). To get (8.1), apply the map ϕ from Lemma 4.3 to each side of (8.2). \square

Proposition 8.2. *For $i \in \mathbb{Z}_4$, the following relations hold on each nonzero finite-dimension \square_q -module:*

$$(8.3) \quad \exp_q(\mathbf{n}_{i,i+1})^{-1}x_i\exp_q(\mathbf{n}_{i,i+1}) = x_ix_{i+1}x_i,$$

$$(8.4) \quad \exp_q(\mathbf{n}_{i,i+1})x_{i+1}\exp_q(\mathbf{n}_{i,i+1})^{-1} = x_{i+1}x_ix_{i+1}.$$

Proof. We first verify (8.3). By (3.8) the element x_ix_{i+1} commutes with $\mathbf{n}_{i,i+1}$. Therefore $\exp_q(\mathbf{n}_{i,i+1})^{-1}x_ix_{i+1}\exp_q(\mathbf{n}_{i,i+1}) = x_ix_{i+1}$ in view of Definition 7.1. Combine this equation with (8.2) to get (8.3). To get (8.4), apply the map ϕ from Lemma 4.3 to each side of (8.3). \square

9. SOME q -EXPONENTIAL FORMULAS, PART II

In this section, we analyze (7.3) for the case $j = i + 2$ or $j = i + 3$.

Lemma 9.1. *For $i \in \mathbb{Z}_4$, the following relations hold in \square_q :*

$$(9.1) \quad \sum_{m=0}^3 (-1)^m q^{3-2m} \frac{\mathbf{n}_{i,i+1}^{3-m}}{[3-m]_q!} x_{i+2} \frac{\mathbf{n}_{i,i+1}^m}{[m]_q!} = -(q - q^{-1})^2 \mathbf{n}_{i,i+1} x_i \mathbf{n}_{i,i+1},$$

$$(9.2) \quad \sum_{m=0}^3 (-1)^m q^{3-2m} \frac{\mathbf{n}_{i,i+1}^m}{[m]_q!} x_{i+3} \frac{\mathbf{n}_{i,i+1}^{3-m}}{[3-m]_q!} = -(q - q^{-1})^2 \mathbf{n}_{i,i+1} x_{i+1} \mathbf{n}_{i,i+1}.$$

Proof. To verify (9.1) let Θ denote the left-hand side of (9.1) minus the right-hand side of (9.1). We will show that $\Theta = 0$. To do this we first eliminate each occurrence of $\mathbf{n}_{i,i+1}$ in Θ using the second equality in (3.6). In the resulting equation, we simplify things using the following principle: for each occurrence of x_{i+1} , move it to the far left using (3.1). The above simplification yields the following results.

The expression $q^3(q - q^{-1})^3 \mathbf{n}_{i,i+1}^3 x_{i+2}$ is a weighted sum involving the following terms and coefficients:

term	x_{i+2}	$x_{i+1} x_i x_{i+2}$	$x_{i+1}^2 x_i^2 x_{i+2}$	$x_{i+1}^3 x_i^3 x_{i+2}$
coeff.	1	$-q^{-2}[3]_q$	$q^{-4}[3]_q$	$-q^{-6}$

The expression $q^3(q - q^{-1})^3 \mathbf{n}_{i,i+1}^2 x_{i+2} \mathbf{n}_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

term	x_{i+2}	x_i	$x_{i+1} x_i x_{i+2}$	$x_{i+1} x_{i+2} x_i$	$x_{i+1} x_i^2$
coeff.	1	$q^2 - 1$	$-1 - q^{-2}$	$-q^{-2}$	$q^{-2} - q^2$

term	$x_{i+1}^2 x_i^2 x_{i+2}$	$x_{i+1}^2 x_i x_{i+2} x_i$	$x_{i+1}^2 x_i^3$	$x_{i+1}^3 x_i^2 x_{i+2} x_i$
coeff.	q^{-2}	$q^{-2} + q^{-4}$	$1 - q^{-2}$	$-q^{-4}$

The expression $q^3(q - q^{-1})^3 \mathbf{n}_{i,i+1} x_{i+2} \mathbf{n}_{i,i+1}^2$ is a weighted sum involving the following terms and coefficients:

term	x_{i+2}	x_i	$x_{i+1} x_i x_{i+2}$	$x_{i+1} x_{i+2} x_i$	$x_{i+1} x_i^2$
coeff.	1	$q^2 - q^{-2}$	-1	$-1 - q^{-2}$	$q^{-1}(q^{-1} + q)(q^{-2} - q^2)$

term	$x_{i+1}^2 x_{i+2} x_i^2$	$x_{i+1}^2 x_i x_{i+2} x_i$	$x_{i+1}^2 x_i^3$	$x_{i+1}^3 x_i x_{i+2} x_i^2$
coeff.	q^{-2}	$q^{-2} + 1$	$1 - q^{-4}$	$-q^{-2}$

The expression $q^3(q - q^{-1})^3 x_{i+2} \mathbf{n}_{i,i+1}^3$ is a weighted sum involving the following terms and coefficients:

term	x_{i+2}	x_i	$x_{i+1} x_{i+2} x_i$	$x_{i+1} x_i^2$
coeff.	1	$q^2 - q^{-4}$	$-[3]_q$	$q^{-2}(q^{-2} - q^2)[3]_q$

term	$x_{i+1}^2 x_{i+2} x_i^2$	$x_{i+1}^2 x_i^3$	$x_{i+1}^3 x_{i+2} x_i^3$
coeff.	$[3]_q$	$1 - q^{-6}$	-1

The expression $q^4(q - q^{-1})^2 \mathbf{n}_{i,i+1} x_i \mathbf{n}_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

term	x_i	$x_{i+1}x_i^2$	$x_{i+1}^2x_i^3$
coeff.	1	$-1 - q^{-2}$	q^{-2}

By the above comments Θ is equal to

$$(9.3) \quad \frac{-x_{i+1}^3(x_i^3x_{i+2} - [3]_q x_i^2x_{i+2}x_i + [3]_q x_i x_{i+2}x_i^2 - x_{i+2}x_i^3)}{q^6(q - q^{-1})^3[3]_q!}.$$

The expression (9.3) is 0 by (3.2). Therefore $\Theta = 0$. We have shown (9.1). To get (9.2) apply the map ϕ from Lemma 4.3 to each side of (9.1). \square

Lemma 9.2. *For $i \in \mathbb{Z}_4$ and $m \in \mathbb{N}$, the following relations hold in \square_q :*

$$(9.4) \quad \begin{aligned} x_{i+2} \mathbf{n}_{i,i+1}^m &= a_m(q) \mathbf{n}_{i,i+1}^m x_{i+2} + b_m(q) \mathbf{n}_{i,i+1}^{m-1} x_{i+2} \mathbf{n}_{i,i+1} \\ &\quad + c_m(q) \mathbf{n}_{i,i+1}^{m-2} x_{i+2} \mathbf{n}_{i,i+1}^2 + d_m(q) \mathbf{n}_{i,i+1}^{m-1} x_i, \end{aligned}$$

$$(9.5) \quad \begin{aligned} \mathbf{n}_{i,i+1}^m x_{i+3} &= a_m(q) x_{i+3} \mathbf{n}_{i,i+1}^m + b_m(q) \mathbf{n}_{i,i+1} x_{i+3} \mathbf{n}_{i,i+1}^{m-1} \\ &\quad + c_m(q) \mathbf{n}_{i,i+1}^2 x_{i+3} \mathbf{n}_{i,i+1}^{m-2} + d_m(q) x_{i+1} \mathbf{n}_{i,i+1}^{m-1}, \end{aligned}$$

$$(9.6) \quad \begin{aligned} \mathbf{n}_{i,i+1}^m x_{i+2} &= a_m(q^{-1}) x_{i+2} \mathbf{n}_{i,i+1}^m + b_m(q^{-1}) \mathbf{n}_{i,i+1} x_{i+2} \mathbf{n}_{i,i+1}^{m-1} \\ &\quad + c_m(q^{-1}) \mathbf{n}_{i,i+1}^2 x_{i+2} \mathbf{n}_{i,i+1}^{m-2} - d_m(q^{-1}) x_i \mathbf{n}_{i,i+1}^{m-1}, \end{aligned}$$

$$(9.7) \quad \begin{aligned} x_{i+3} \mathbf{n}_{i,i+1}^m &= a_m(q^{-1}) \mathbf{n}_{i,i+1}^m x_{i+3} + b_m(q^{-1}) \mathbf{n}_{i,i+1}^{m-1} x_{i+3} \mathbf{n}_{i,i+1} \\ &\quad + c_m(q^{-1}) \mathbf{n}_{i,i+1}^{m-2} x_{i+3} \mathbf{n}_{i,i+1}^2 - d_m(q^{-1}) \mathbf{n}_{i,i+1}^{m-1} x_{i+1}, \end{aligned}$$

where

$$a_m(q) = q^{2m} \frac{[m-1]_q [m-2]_q}{[2]_q},$$

$$b_m(q) = -q^{2m-2} [m]_q [m-2]_q,$$

$$c_m(q) = q^{2m-4} \frac{[m]_q [m-1]_q}{[2]_q},$$

$$d_m(q) = q^{m-5} [3m]_q - q^{-3} [3]_q [2m]_q + q^{-m-1} [3]_q [m]_q.$$

Proof. To get (9.4), use (3.2), (9.1) and induction on m . To get (9.5), apply the map ϕ from Lemma 4.3 to each side of (9.4). Concerning (9.6), apply the map \dagger from Lemma 4.7 to each side of (9.4). This yields an equation that holds in $\square_{q^{-1}}$. In this equation replace q by q^{-1} . This gives (9.6). To get (9.7), apply ϕ to each side of (9.6). \square

We now analyze (7.3) for the case $j = i + 3$.

Proposition 9.3. *For $i \in \mathbb{Z}_4$, the following relation holds on each nonzero finite-dimensional \square_q -module:*

$$\begin{aligned} \exp_q(\mathbf{n}_{i,i+1})^{-1} x_{i+3} \exp_q(\mathbf{n}_{i,i+1}) &= x_{i+3} - x_i^{-1} + \frac{q x_i x_{i+1} x_{i+3}}{q - q^{-1}} - \frac{x_i x_{i+3} x_{i+1}}{q(q - q^{-1})} \\ &\quad + \frac{q^3 x_i^2 x_{i+1}^2 x_{i+3}}{(q - q^{-1})(q^2 - q^{-2})} + \frac{q x_i^2 x_{i+3} x_{i+1}^2}{(q - q^{-1})(q^2 - q^{-2})} - \frac{q^2 x_i^2 x_{i+1} x_{i+3} x_{i+1}}{(q - q^{-1})^2}. \end{aligned}$$

Proof. For $m \in \mathbb{N}$ multiply each side of (9.7) by $q^{\binom{m}{2}}/[m]_q!$. Sum the resulting equations over $m \in \mathbb{N}$ and evaluate the result using (7.1) to get

$$\begin{aligned} x_{i+3}\exp_q(\mathbf{n}_{i,i+1}) &= \frac{q^3\exp_q(q^{-4}\mathbf{n}_{i,i+1})x_{i+3}}{(q-q^{-1})(q^2-q^{-2})} - \frac{\exp_q(q^{-2}\mathbf{n}_{i,i+1})x_{i+3}}{(q-q^{-1})^2} \\ &\quad + \frac{\exp_q(\mathbf{n}_{i,i+1})x_{i+3}}{q^3(q-q^{-1})(q^2-q^{-2})} - \frac{\exp_q(\mathbf{n}_{i,i+1})x_{i+3}\mathbf{n}_{i,i+1}}{q(q-q^{-1})} \\ &\quad + \frac{q\exp_q(q^{-2}\mathbf{n}_{i,i+1})x_{i+3}\mathbf{n}_{i,i+1}}{q-q^{-1}} + \frac{q\exp_q(\mathbf{n}_{i,i+1})x_{i+3}\mathbf{n}_{i,i+1}^2}{q+q^{-1}} \\ &\quad - \exp_q(q^2\mathbf{n}_{i,i+1})x_{i+1} + (1+q^2)\exp_q(\mathbf{n}_{i,i+1})x_{i+1} \\ &\quad - q^2\exp_q(q^{-2}\mathbf{n}_{i,i+1})x_{i+1}. \end{aligned}$$

In the above equation multiply each term on the left by $\exp_q(\mathbf{n}_{i,i+1})^{-1}$ and use (7.4) to get that $\exp_q(\mathbf{n}_{i,i+1})^{-1}x_{i+3}\exp_q(\mathbf{n}_{i,i+1})$ is equal to

$$\begin{aligned} &\frac{q^3x_i^2x_{i+1}^2x_{i+3}}{(q-q^{-1})(q^2-q^{-2})} - \frac{x_ix_{i+1}x_{i+3}}{(q-q^{-1})^2} + \frac{x_{i+3}}{q^3(q-q^{-1})(q^2-q^{-2})} \\ &- \frac{x_{i+3}\mathbf{n}_{i,i+1}}{q(q-q^{-1})} + \frac{qx_ix_{i+1}x_{i+3}\mathbf{n}_{i,i+1}}{q-q^{-1}} + \frac{qx_{i+3}\mathbf{n}_{i,i+1}^2}{q+q^{-1}} \\ &- x_i^{-1} + (1+q^2)x_{i+1} - q^2x_ix_{i+1}^2. \end{aligned}$$

For notational convenience let Ψ denote the above expression. In Ψ we first eliminate every occurrence of $\mathbf{n}_{i,i+1}$ using the second equality in (3.6). In the resulting expression, we simplify things using the following principle: for each occurrence of x_i , move it to the far left using (3.1). The above simplification yields the following results.

The expression $-q^{-1}(q-q^{-1})^{-1}x_{i+3}\mathbf{n}_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

term	x_{i+3}	$x_ix_{i+3}x_{i+1}$	x_{i+1}
coeff.	$-(q-q^{-1})^{-2}$	$q^{-2}(q-q^{-1})^{-2}$	$q^{-1}(q-q^{-1})^{-1}$

The expression $q(q-q^{-1})^{-1}x_ix_{i+1}x_{i+3}\mathbf{n}_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

term	$x_ix_{i+1}x_{i+3}$	$x_i^2x_{i+1}x_{i+3}x_{i+1}$	$x_ix_{i+3}x_{i+1}$	$x_ix_{i+1}^2$
coeff.	$q^2(q-q^{-1})^{-2}$	$-q^2(q-q^{-1})^{-2}$	$q(q-q^{-1})^{-1}$	$-q(q-q^{-1})^{-1}$

The expression $q(q+q^{-1})^{-1}x_{i+3}\mathbf{n}_{i,i+1}^2$ is a weighted sum involving the following terms and coefficients:

term	x_{i+3}	$x_ix_{i+3}x_{i+1}$	x_{i+1}
coeff.	$q^3(q-q^{-1})^{-1}(q^2-q^{-2})^{-1}$	$-q^2(q-q^{-1})^{-2}$	$-q^3(q-q^{-1})^{-1}$

term	$x_i^2x_{i+3}x_{i+1}^2$	$x_ix_{i+1}^2$
coeff.	$q(q-q^{-1})^{-1}(q^2-q^{-2})^{-1}$	$q^3(q-q^{-1})^{-1}$

Evaluating Ψ using the above comments, we get the result. \square

Proposition 9.4. *For $i \in \mathbb{Z}_4$, the following relation holds on each nonzero finite-dimensional \square_q -module:*

$$\begin{aligned} \exp_q(\mathbf{n}_{i,i+1})x_{i+3}\exp_q(\mathbf{n}_{i,i+1})^{-1} &= x_{i+1} - \frac{x_{i+3}}{(q - q^{-1})^2} + \frac{qx_{i+1}^{-1}x_{i+3}x_{i+1}}{(q - q^{-1})(q^2 - q^{-2})} \\ &\quad + \frac{q^{-1}x_{i+1}x_{i+3}x_{i+1}^{-1}}{(q - q^{-1})(q^2 - q^{-2})}. \end{aligned}$$

Proof. For $m \in \mathbb{N}$ multiply each side of (9.5) by $q^{-2m}q^{\binom{m}{2}}/[m]_q!$. Sum the resulting equations over $m \in \mathbb{N}$ and evaluate the result using (7.1) to get

$$\begin{aligned} \exp_q(q^{-2}\mathbf{n}_{i,i+1})x_{i+3} &= \frac{x_{i+3}\exp_q(q^2\mathbf{n}_{i,i+1})}{q^3(q - q^{-1})(q^2 - q^{-2})} - \frac{x_{i+3}\exp_q(\mathbf{n}_{i,i+1})}{(q - q^{-1})^2} \\ &\quad + \frac{q^3x_{i+3}\exp_q(q^{-2}\mathbf{n}_{i,i+1})}{(q - q^{-1})(q^2 - q^{-2})} + \frac{\mathbf{n}_{i,i+1}x_{i+3}\exp_q(\mathbf{n}_{i,i+1})}{q(q - q^{-1})} \\ &\quad - \frac{\mathbf{n}_{i,i+1}x_{i+3}\exp_q(q^2\mathbf{n}_{i,i+1})}{q^3(q - q^{-1})} + \frac{\mathbf{n}_{i,i+1}^2x_{i+3}\exp_q(q^2\mathbf{n}_{i,i+1})}{q^3(q + q^{-1})} \\ &\quad + \frac{x_{i+1}\exp_q(q^{-2}\mathbf{n}_{i,i+1})}{q^2} - \frac{(q + q^{-1})x_{i+1}\exp_q(\mathbf{n}_{i,i+1})}{q^3} \\ &\quad + \frac{x_{i+1}\exp_q(q^2\mathbf{n}_{i,i+1})}{q^4}. \end{aligned}$$

In the above equation multiply each term on the left by $x_{i+1}^{-1}x_i^{-1}$ and on the right by $\exp_q(\mathbf{n}_{i,i+1})^{-1}$, and then use (7.5) to get that $\exp_q(\mathbf{n}_{i,i+1})x_{i+3}\exp_q(\mathbf{n}_{i,i+1})^{-1}$ is equal to

$$\begin{aligned} &\frac{x_{i+1}^{-1}x_i^{-1}x_{i+3}x_i^{-1}x_{i+1}}{q^3(q - q^{-1})(q^2 - q^{-2})} - \frac{x_{i+1}^{-1}x_i^{-1}x_{i+3}}{(q - q^{-1})^2} + \frac{q^3x_{i+1}^{-1}x_i^{-1}x_{i+3}x_ix_{i+1}}{(q - q^{-1})(q^2 - q^{-2})} \\ &\quad + \frac{x_{i+1}^{-1}x_i^{-1}\mathbf{n}_{i,i+1}x_{i+3}}{q(q - q^{-1})} - \frac{x_{i+1}^{-1}x_i^{-1}\mathbf{n}_{i,i+1}x_{i+3}x_i^{-1}x_{i+1}}{q^3(q - q^{-1})} \\ &\quad + \frac{x_{i+1}^{-1}x_i^{-1}\mathbf{n}_{i,i+1}^2x_{i+3}x_i^{-1}x_{i+1}}{q^3(q + q^{-1})} + \frac{x_{i+1}^{-1}x_i^{-1}x_{i+1}x_ix_{i+1}}{q^2} \\ &\quad - \frac{(q + q^{-1})x_{i+1}^{-1}x_i^{-1}x_{i+1}}{q^3} + \frac{x_{i+1}^{-1}x_i^{-1}x_{i+1}x_i^{-1}x_{i+1}}{q^4}. \end{aligned}$$

For notational convenience let Φ denote the above expression. In Φ we first eliminate every occurrence of $\mathbf{n}_{i,i+1}$ using the first equality in (3.6). In the resulting expression, we simplify things using (3.1), (5.1), and (5.2). Our guiding principle is to bring x_i, x_i^{-1} together for cancellation, and also to bring x_{i+1}, x_{i+1}^{-1} together for cancellation. The above simplification yields the following results.

The expression $q^3(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1}x_{i+1}^{-1}x_i^{-1}x_{i+3}x_ix_{i+1}$ is a weighted sum involving the following terms and coefficients:

term	$x_{i+1}^{-1}x_{i+3}x_{i+1}$	x_i^{-1}	$x_{i+1}^{-1}x_i^{-2}$
coeff.	$q(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1}$	$q^4(q^2 - q^{-2})^{-1}$	$-q^3(q + q^{-1})^{-1}$

The expression $q^{-1}(q - q^{-1})^{-1}x_{i+1}^{-1}x_i^{-1}\mathbf{n}_{i,i+1}x_{i+3}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3}}{(q - q^{-1})^{-2}} \quad \frac{x_{i+3}}{-(q - q^{-1})^{-2}} \right.$$

The expression $-q^{-3}(q - q^{-1})^{-1} x_{i+1}^{-1} x_i^{-1} \mathbf{n}_{i,i+1} x_{i+3} x_i^{-1} x_{i+1}^{-1}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{-q^{-2}(q - q^{-1})^{-2}} \quad \frac{x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q^{-2}(q - q^{-1})^{-2}} \right.$$

The expression $q^{-3}(q + q^{-1})^{-1} x_{i+1}^{-1} x_i^{-1} \mathbf{n}_{i,i+1}^2 x_{i+3} x_i^{-1} x_{i+1}^{-1}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q^{-1}(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1}} \quad \frac{x_{i+3} x_i^{-1} x_{i+1}^{-1}}{-q^{-2}(q - q^{-1})^{-2}} \right.$$

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_i^{-1}}{-q^{-4}(q^2 - q^{-2})^{-1}} \quad \frac{x_i^{-2} x_{i+1}^{-1}}{-q^{-3}(q + q^{-1})^{-1}} \quad \frac{x_{i+1} x_{i+3} x_{i+1}^{-1}}{q^{-1}(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1}} \right.$$

The expression $q^{-2} x_{i+1}^{-1} x_i^{-1} x_{i+1} x_i x_{i+1}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_{i+1}}{1} \quad \frac{x_i^{-1}}{1 - q^2} \quad \frac{x_{i+1}^{-1} x_i^{-2}}{(q - q^{-1})^2} \right.$$

The expression $-q^{-3}(q + q^{-1}) x_{i+1}^{-1} x_i^{-1} x_{i+1}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_i^{-1}}{-q^{-1}(q + q^{-1})} \quad \frac{x_{i+1}^{-1} x_i^{-2}}{q^{-2}(q^2 - q^{-2})} \right.$$

Using (5.3) the expression $q^{-4} x_{i+1}^{-1} x_i^{-1} x_{i+1} x_i^{-1} x_{i+1}^{-1}$ is a weighted sum involving the following terms and coefficients:

$$\frac{\text{term}}{\text{coeff.}} \left| \frac{x_i^{-2} x_{i+1}^{-1}}{q^{-3}(q + q^{-1})^{-1}} \quad \frac{x_{i+1}^{-1} x_i^{-2}}{q^{-5}(q + q^{-1})^{-1}} \right.$$

Evaluating Φ using the above comments we get the result. \square

We now analyze (7.3) for the case $j = i + 2$.

Proposition 9.5. *For $i \in \mathbb{Z}_4$, the following relation holds on each nonzero finite-dimensional \square_q -module:*

$$\begin{aligned} \exp_q(\mathbf{n}_{i,i+1})^{-1} x_{i+2} \exp_q(\mathbf{n}_{i,i+1}) &= x_i - \frac{x_{i+2}}{(q - q^{-1})^2} + \frac{q x_i x_{i+2} x_i^{-1}}{(q - q^{-1})(q^2 - q^{-2})} \\ &\quad + \frac{q^{-1} x_i^{-1} x_{i+2} x_i}{(q - q^{-1})(q^2 - q^{-2})}. \end{aligned}$$

Proof. Apply the map ϕ from Lemma 4.3 to each side of the equation in Proposition 9.4. \square

Proposition 9.6. *For $i \in \mathbb{Z}_4$, the following relation holds on each nonzero finite-dimensional \square_q -module:*

$$\begin{aligned} \exp_q(\mathbf{n}_{i,i+1})x_{i+2}\exp_q(\mathbf{n}_{i,i+1})^{-1} &= x_{i+2} - x_{i+1}^{-1} + \frac{qx_{i+2}x_ix_{i+1}}{q - q^{-1}} - \frac{x_ix_{i+2}x_{i+1}}{q(q - q^{-1})} \\ &+ \frac{q^3x_{i+2}x_i^2x_{i+1}^2}{(q - q^{-1})(q^2 - q^{-2})} + \frac{qx_i^2x_{i+2}x_{i+1}^2}{(q - q^{-1})(q^2 - q^{-2})} - \frac{q^2x_ix_{i+2}x_ix_{i+1}^2}{(q - q^{-1})^2}. \end{aligned}$$

Proof. Apply the map ϕ from Lemma 4.3 to each side of the equation in Proposition 9.3. \square

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